Ergodic Theory - Week 1

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1 Measure Preserving Systems

P1. Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $T : X \to X$ be a measurable map. Show that T preserves μ if and only if

$$\int f \circ T \, d\mu = \int f d\mu \tag{1}$$

holds for any $f \in L^1(X)$.

Optional: If X is a Polish space (Hausdorff completely metrizable topological space) and μ is a Borel probability measure on X show that it suffices to check (1) for any $f \in C(X)$.

Hint: Use the fact that any finite Borel measure on a Polish space is regular, namely for any measurable set A and any $\varepsilon > 0$ there exist open U and compact K such that $K \subseteq A \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.

Assume that T preserves μ. Then (1) holds for indicator functions. Indeed, for any $A \in \mathcal{A}$,

$$\int \mathbb{1}_A \circ T \, d\mu = \int \mathbb{1}_{T^{-1}A} \, d\mu = \mu(T^{-1}A) = \mu(A) = \int \mathbb{1}_A \, d\mu.$$

Consequently, (1) holds for simple functions, i.e. functions of the form $f = \sum_{i=1}^{k} a_i \mathbb{1}_{A_i}$, where $k \in \mathbb{N}, a_i \in \mathbb{C}$ and $A_i \in \mathcal{A}$ for any $i = 1, \ldots, k$. Now let $f \in L^1(X)$. Then there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ converging to f in $L^1(X)$. Then

$$\int f \circ T d\mu = \lim_{n\to\infty} \int f_n \circ T d\mu = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Conversely, assume that (1) holds for any $f \in L^1(X)$. Then for any $A \in \mathcal{A}$, applying (1) for $f = \mathbb{1}_A$ yields $\mu(T^{-1}A) = \mu(A)$, showing that T preserves μ. Now suppose that holds for any $f \in L^1(X)$.

Now, suppose X is a Polish space and assume that (1) holds for continuous functions, and let $f = \mathbb{1}_A \in L^1(X)$. Since $C(X)$ is dense in $L^1(X)$, there exists a sequence $(f_n)_n \in \mathbb{N}$ of continuous functions converging to f. Moreover, this sequence can be taken bounded, given that thanks to the regularity of μ (given that X is Polish), we can always find a closed set F and an open set U such that $F \subseteq A \subseteq E$ and $\mu(U \setminus F) < 1/n$, thus we can take

$$f_n(x) = \frac{d(x, U^c)}{d(x, U^c) + d(x, F)},$$

which is such that $||f_n - f||_1 < 1/n$ for every n. Then, using the Dominated Convergence

Theorem:

$$\int f \circ T d\mu = \int \lim_{n\to\infty} f_n \circ T d\mu = \lim_{n\to\infty} \int f_n \circ T d\mu = \lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu = \int f d\mu.$$

P2. Let $(X_1, \mathcal{A}_1, \mu_1, T_1)$ and $(X_2, \mathcal{A}_2, \mu_2, T_2)$ be measure preserving systems. Prove that the product system $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2, T_1 \times T_2)$ is also measure-preserving.

Observe that for any sets $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$, we have

$$(\mu_1 \times \mu_2)((T_1 \times T_2)^{-1}(A_1 \times A_2)) = \mu_1(T_1^{-1}A_1)\mu_2(T_2^{-1}A_2) = \mu_1(A_1)\mu_2(A_2) = (\mu_1 \times \mu_2)(A_1 \times A_2).$$

The σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ is generated by the algebra of the sets $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$ and in this algebra, the measure $\mu_1 \times \mu_2$ and the pushforward measure $(T_1 \times T_2)(\mu_1 \times \mu_2)$ defined by $(T_1 \times T_2)(\mu_1 \times \mu_2)(A) = (\mu_1 \times \mu_2)((T_1 \times T_2)^{-1}(A))$ take identical values. Thus, by Carathéodory's Theorem, we have that these measures are equal on $\mathcal{A}_1 \otimes \mathcal{A}_2$ which establishes our claim.

P3. We consider the torus with the Borel σ-algebra and the Lebesgue measure.

(a) Show that for any $a \in \mathbb{R}$, the map $Tx = x + a \ (mod\, 1)$ preserves the Lebesgue measure.

We will use the first exercise for this. It suffices to show that for any continuous function $f : \mathbb{T} \to \mathbb{C}$ we have

$$\int_{\mathbb{T}} f(Tx)\ d\lambda(x) = \int_{\mathbb{T}} f(x)\ d\lambda(x).$$

We can extend f periodically modulo 1, so that f is a 1-periodic continuous function. The previous equality can be rewritten as

$$\int_0^1 f(x+a)\ dx = \int_0^1 f(x)\ dx.$$

However,

$$\int_0^1 f(x+a)\ dx = \int_a^{1+a} f(x)\ dx = \int_a^1 f(x)\ dx + \int_1^{1+a} f(x)\ dx = \int_a^1 f(x)\ dx + \int_0^a f(x)\ dx = \int_0^1 f(x)\ dx$$

using the 1-periodicity of f.

(b) For each $p \in \mathbb{N}$ we define the map $T_p x = px \ (mod\, 1)$ for all $x \in [0, 1)$. Show that the transformation T_p preserves the Lebesgue measure.

In view of the first exercise, it suffices to show that for any continuous function $f : \mathbb{T} \to \mathbb{C}$ we have

$$\int_{\mathbb{T}} f \circ T_p\ d\lambda = \int f\ d\lambda.$$

The function f is 1-periodic so we have

$$\int_0^1 f(px)\ d\lambda(x) = \int_0^{\frac{1}{p}} f(px)\ dx + \cdots + \int_{\frac{p-1}{p}}^1 f(px)\ dx =$$
$$\frac{1}{p}\int_0^1 f(x)\ dx + \cdots + \frac{1}{p}\int_{p-1}^p f(x)\ dx = \frac{1}{p}\int_0^1 f(x)dx + \cdots + \frac{1}{p}\int_0^1 f(x)dx = \int_0^1 f(x)dx,$$

where we used the 1-periodicity of f in the penultimate step.

P4. Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \to X$ be an invertible measure preserving transformation (with respect to μ). Now, at any moment, instead of moving forward by T (that is, instead of looking at the map $x \to Tx$), we flip a fair coin to decide whether we will use T or T^{-1}.

The goal is to describe the random system described above by means of a measure-preserving system. In particular, we want to find a map R such that given a point x and a sequence of coin tosses ω, we would have that $R(\omega, x)$ would produce the same result as the procedure above.

(a) Find a probability space $(Y, \mathcal{B}, v)$ and a measure preserving map S that models the sequence of coin tosses.

Consider $\Omega = \{0,1\}^{\mathbb{N}}$ with the Borel σ-algebra $\mathcal{B}$ generated by cylinder sets and the uniform product measure v. Thus, for any finite string $(a_1, \dots, a_k)$ of 0's and 1's, we have

$$v\left(\{\omega \in \Omega : \omega_1 = a_1 \dots \omega_k = a_k\}\right) = \frac{1}{2^k}.$$

We endow this space with the left-shift S (that is $S\omega$ is the sequence such that $S\omega(n) = \omega(n+1)$ for all $n \in \mathbb{N}$), which preserves the measure v. This models the sequence of fair coin tosses. Indeed, for any element ω of the sample space (i.e. a sequence of coin tosses), the first coordinate of $S^n\omega$ is the outcome of the n-th coin toss.

(b) Consider the product system $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times v)$. Define a measure-preserving map R on this product space that models the original random system.

We consider the product space $Y = \Omega \times X$ with the product σ-algebra $\mathcal{B} \times \mathcal{A}$ and the product measure $\lambda = v \times \mu$. On this space, we define the map $R(\omega, x) = (S\omega, T^{2\omega_1 - 1})$, where ω_1 is the first coordinate of ω. We will prove that this transformation preserves the measure λ and it models the random walk system described in the exercise. Indeed, given any sequence of coin tosses ω, the transformation R checks the first coordinate of ω and applies T if $\omega_1 = 1$ and T^{-1} if $\omega_1 = 0$. Then, it shifts the sequence ω to the next coordinate (the next coin toss).

We prove that R preserves the measure λ. We will only need to prove that this holds for sets of the form $B \times A$, where $B \in \mathcal{B}$ and $A \in \mathcal{A}$, since these generate the product

σ-algebra. Then, we have

$$\begin{aligned}
\lambda(R^{-1}(B \times A)) &= \lambda\left(\{R(\omega, x) \in B \times A\}\right) \\
&= \lambda\left(\{R(\omega, x) \in B \times A, \omega_1 = 0\} \cup \{R(\omega, x) \in B \times A, \omega_1 = 1\}\right) \\
&= \lambda\left(\{R(\omega, x) \in B \times A, \omega_1 = 0\}\right) + \lambda\left(\{R(\omega, x) \in B \times A, \omega_1 = 1\}\right) \\
&= \lambda\left((\omega, x) \colon \omega_1 = 0, S\omega \in B, T^{-1}x \in A\right) + \lambda\left((\omega, x) \colon \omega_1 = 1, S\omega \in B, Tx \in A\right) \\
&= \lambda\left(\left(\{\omega_1 = 0\} \cap S^{-1}B\right) \times T(A)\right) + \lambda\left(\left(\{\omega_1 = 0\} \cap S^{-1}B\right) \times T^{-1}(A)\right) \\
&= v\left(\left(\{\omega_1 = 0\} \cap S^{-1}B\right)\right) \mu(T(A)) + v\left(\left(\{\omega_1 = 1\} \cap S^{-1}B\right)\right) \mu(T^{-1}(A)) \\
&= v\left(\left(\{\omega_1 = 0\} \cap S^{-1}B\right)\right) \mu(A) + v\left(\left(\{\omega_1 = 1\} \cap S^{-1}B\right)\right) \mu(A) \\
&= v(S^{-1}B)\mu(A) = v(B)\mu(A) = \lambda(B \times A),
\end{aligned}$$

where we used the fact that $\mu(TA) = \mu(T^{-1}A) = \mu(A)$ because T, T^{-1} are measure preserving.

P5. A set $R \subseteq \mathbb{Z}$ is a set of recurrence if for every measure-preserving system $(X, \mathcal{A}, \mu, T)$ and for all $A \in \mathcal{A}$ with $\mu(A) > 0$ we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in R \setminus \{0\}$.

(a) Show that $2\mathbb{N}$ is a set of recurrence but $2\mathbb{N} + 1$ is not.

Let $(X, \mathcal{A}, \mu, T)$ be any measure preserving system. Consider the system $(X, \mathcal{A}, \mu, T^2)$ which is also measure preserving and for which Poincaré's Recurrence Theorem asserts that $\mathbb{N}$ is a set of recurrence for this system. Thus, there is some $n \in \mathbb{N}$ such that $\mu(A \cap T^{-2n}A) > 0$ or equivalently there is $n \in 2\mathbb{N}$ such that $\mu(A \cap T^{-n}A) > 0$, concluding that $2\mathbb{N}$ is a set of recurrence.

To see that $2\mathbb{N}+1$ is not a set of recurrence, consider the rotation on two points. Namely, let $X = \{0, 1\}$ with sigma algebra $\mathcal{A} = \mathcal{P}(X)$ and measure $\mu(\{0\}) = \mu(\{1\}) = 1/2$ and equip it with the transformation S which brings 0 to 1 and vice versa. Notice that for all $n \in 2\mathbb{N} + 1$ we have that $\{0\} \cap T^{-n}\{0\} = \emptyset$. Therefore, $2\mathbb{N} + 1$ cannot be a set of recurrence.

(b) Show that sets of recurrence possess the Ramsey property: if $R = R_1 \cup \ldots \cup R_k$ is a set of recurrence, then one of the sets $R_1, \ldots, R_k$ is also a set of recurrence.

We prove this in the case $k = 2$. The general case follows easily by induction. Suppose by contradiction we have a set of recurrence $R = R_1 \cup R_2$ such that neither R_1 nor R_2 is a set of recurrence. Then for every $i \in \{1, 2\}$ there exist a system $(X_i, \mathcal{A}_i, \mu_i, T_i)$ and a set $A_i \in \mathcal{A}_i$, with $\mu(A_i \cap T_i^{-n}A_i) = 0$, for all n$\in R_i \setminus \{0\}$. Now consider the product system $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2, T_1 \times T_2)$ and notice that $\mu_1 \otimes \mu_2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) > 0$. Therefore, as R is a set of recurrence there exists $n \in R \setminus \{0\}$ such that

$$0 < \mu_1 \otimes \mu_2(A_1 \times A_2 \cap (T_1 \times T_2)^{-n} A_1 \times A_2) = \mu_1(A_1 \cap T_1^{-n} A_1)\mu_2(A_2 \cap T_2^{-n} A_2) = 0,$$

where the last equality comes from the fact that $n \in R = R_1 \cup R_2$. This is a contradiction, hence either R_1 or R_2 is a set of recurrence.

(c) Show that if $R \subseteq \mathbb{Z}$ is a set of recurrence, then so is $R \cap m\mathbb{Z}$ for every $m \in \mathbb{Z}$.

Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system, $A \in \mathcal{A}$ with $\mu(A) > 0$, $R \subseteq \mathbb{Z}$ a set of recurrence, and $m \in \mathbb{Z}$. To prove that $R \cap m\mathbb{Z}$ is a set of recurrence, we consider the auxiliary system $(\mathbb{Z}_m, P(\mathbb{Z}_m), \nu, S)$, where $\mathbb{Z}_m = \{0, \ldots, m-1\}$, ν is the uniform measure and S is the rotation $Sx = x + 1 \mod m$ for every $x \in \mathbb{Z}_m$. Now, we consider the product system $(X \times \mathbb{Z}_m, \mathcal{A} \otimes P(\mathbb{Z}_m), \mu \otimes \nu, T \times S)$. Given that R is a set of recurrence, there exists $n \in R \setminus \{0\}$ such that

$$\mu \otimes \nu(A \times \{0\} \cap (T \times S)^{-n} A \times \{0\}) > 0.$$

Observe that the left-side of the equation is equal to $\mu(A \cap T^{-n}A) \cdot \nu(\{0\} \cap S^{-n}\{0\})$. Thus, $\{0\} \cap S^{-n}\{0\} \neq \emptyset$, which can only be possible if m divides n. Therefore $n \in (R \cap m\mathbb{Z}) \setminus \{0\}$ is such that $\mu(A \cap T^{-n}A) > 0$, which is the desired conclusion.

P6. Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and let $A \in \mathcal{A}$ be a set of positive measure. Define its set of return times

$$R(A) = \{n \in \mathbb{N} \colon \mu(A \cap T^{-n}A) > 0\}.$$

Poincare's recurrence theorem asserts that $R(A)$ is non-empty. In this exercise, we want to show that $R(A)$ is also large in some appropriate sense.

(a) Show that $R(A)$ intersects any difference set: if $E \subset \mathbb{N}$ is infinite, then $R(A) \cap (E - E) \neq \emptyset$. Here, $E - E = \{n \in \mathbb{N} \colon n = a - b \text{ for some } a, b \in E, a > b\}$.

Let $E = \{n_j, j \in \mathbb{N}\}$ be the given set, where n_j is a strictly increasing sequence of integers. Suppose that $R(A) \cap (E - E) = \emptyset$. Then, for any $i < j$, we have $\mu(T^{-n_i}A \cap T^{-n_j}A) = \mu(A \cap T^{-n_j + n_i}A) = 0$. Thus, the sets $\mu(T^{-n_i}A), i \in \mathbb{N}$ are pairwise disjoint (modulo null sets) and have positive measure equal to $\mu(A)$. This contradicts the pigeonhole principle.

(b) Show that the set $R(A)$ has bounded gaps, that is there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have $R(A) \cap \{m, m+1, \ldots, m+k\} \neq 0$.

Assume the conclusion is false, so that we can find arbitrarily long intervals in $\mathbb{N}$ that do not contain any element of $R(A)$. Namely, for any $M \in \mathbb{N}$ we can find (arbitrarily large) $a \in \mathbb{N}$ such that $\{a, a+1, \ldots, a+M\} \cap R(A) = \emptyset$.

The main idea is to construct a sequence n_i inductively so that $R(A)$ does not contain any of the differences $n_i - n_j$, which would contradict (a).

We construct our set E as follows. Let n_1 be an arbitrary integer not in $R(A)$ and put $n_1 \in E$. Using our assumption, we can find $n_2' > n_1$ which satisfy the property $\{n_2', n_2'+1, \ldots, n_2'+n_1\} \cap R(A) = \emptyset$. Then, if we put $n_2 = n_1 + n_2'$ to be the next element in E, we have $n_2 - n_1 \notin R(A)$

Similarly, we find $n_3' > n_2 > n_1$ such that $\{n_3', \ldots, n_3' + n_2\} \cap R(A) = \emptyset$. Thus, if $n_3 = n_2 + n_3'$, our construction gives that $n_3 - n_2, n_3 - n_1$ are not elements of $R(A)$.

Proceeding inductively we construct $n_1 < \ldots < n_k$ such that all pairwise differences of these elements do not belong to $R(A)$. Then, we pick $n_{k+1}' > n_k$ for which we have $\{n_{k+1}', \ldots, n_{k+1}' + n_k\} \cap R(A) = \emptyset$. Then, we define $n_{k+1} = n_k' + n_{k-1}$ to be the next element in our sequence and it is clear that $n_{k+1} - n_j \notin R(A)$ for all $1 \leq i \leq k$. In this way, we construct inductively a sequence n_k so that pairwise differences of this sequence do not belong to $R(A)$. We produce an infinite set E such that $E - E$ has trivial intersection with $R(A)$, which is a contradiction.